Homogeneous Eqns With Constant Coefficients Example

1. Solve $y'' + 2y' - 3y = 0$

Soln:

Let $y = e^{rt}$.

$y' = re^{rt}$ and $y'' = r^2 e^{rt}$

$r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0$

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$r_1 = -3,\ r_2 = 1$

$y_1 = e^{r_1 t} = e^{-3t}$ $\quad$ $y_2 = e^{r_2 t} = e^{t}$

$y = C_1 y_1 + C_2 y_2$
$= C_1 e^{-3t} + C_2 e^{t}$

2. Solve $y'' + 3y' + 2y = 0$

Soln:

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r_1 = -2,\ r_2 = -1$

$y_1 = e^{-2t},\ y_2 = e^{-t}$

$y = C_1 e^{-2t} + C_2 e^{-t}$

3. Solve $6y'' - y' - y = 0$

Soln:

$6r^2 - r - 1 = 0$

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{1 \pm \sqrt{25}}{12}$

$= \dfrac{1 \pm 5}{12}$

$= \frac{1}{2}$ or $-\frac{1}{3}$

$y = C_1 e^{t/2} + C_2 e^{-t/3}$

4. Solve $y'' + 5y' = 0$

Soln:

$r^2 + 5r = 0$

$r(r+5) = 0$

$r_1 = 0, \ r_2 = -5$

$y = C_2 e^{-5t} + C_1$

5. Find the Wronksian of the given pair of functions:

a) $e^{2t}, \ e^{-3t/2}$

Soln:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2} e^{-3t/2} \end{vmatrix}$$

$$= -\frac{3}{2}e^{2t}\cdot e^{-\frac{3t}{2}} - 2e^{2t}\cdot e^{-3t/2}$$

$$= -\frac{3}{2}e^{t/2} - 2e^{t/2}$$

$$= -\frac{7}{2}e^{t/2}$$

$$\neq 0$$

b) $\cos t$, $\sin t$

Soln:

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= (\cos^2 t) + \sin^2 t$$

$$= 1$$

6. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are 2 solns to $t^2 y'' - 2y = 0$ for $t > 0$. Then show that $y = c_1 t^2 + c_2 t^{-1}$ is also a soln of this eqn for any c_1 and c_2.

Soln:

$t^2(t^2)'' - 2(t^2)$

$= 2t^2 - 2t^2$

$= 0$

LHS = RHS

$t^2(t^{-1})'' - 2(t^{-1})$

$= \frac{2}{t} - \frac{2}{t}$

$= 0 \rightarrow$ LHS = RHS

Hence, $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are 2 solns to the given eqn.

$$W = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix}$$

$$= (t^2)(-t^{-2}) - (2t)(t^{-1})$$

$$= -1 - 2$$

$$= -3$$

7. If the Wronksian of f and g is $t\cos t - \sin t$ and if $u = f + 3g$ and $v = f - g$, find the Wronksian of u and v.

Soln

$$W[u,v] = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

$$= uv' - vu'$$
$$= (f+3g)(f-g)' - (f-g)(f+3g)'$$
$$= f(f-g)' + 3g(f-g)' - f(f+3g)' + g(f+3g)'$$
$$= \cancel{f \cdot f'} - fg' + 3gf' - \cancel{3gg'} - \cancel{ff'} - f3g' + gf' + \cancel{g3g'}$$
$$= -fg' + 3gf' - 3fg' + gf'$$
$$= 4gf' - 4fg'$$
$$= 4(gf' - fg')$$
$$= -4\underbrace{(fg' - gf')}_{\text{This is the Wronksian of f and g.}}$$

$$= -4(t\cos t - \sin t)$$

8. Assume that y_1 and y_2 are a fundamental set of solns of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = a_1 y_1 + a_2 y_2$ and $y_4 = b_1 y_1 + b_2 y_2$ where $a_1, a_2, b_1, b2$ are any constants.

a) Show that $W[y_3, y_4] = (a_1 b_2 - a_2 b_1)[W[y_1, y_2]]$

Soln

$$\underset{[y_3,y_4]}{W} = \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix}$$

$$= y_3 y_4' - y_4 y_3'$$
$$= (a_1 y_1 + a_2 y_2)(b_1 y_1 + b_2 y_2)' - (b_1 y_1 + b_2 y_2)(a_1 y_1 + a_2 y_2)'$$
$$= \cancel{a_1 y_1 b_1 y_1'} + a_1 y_1 b_2 y_2' + a_2 y_2 b_1 y_1' + \cancel{a_2 y_2 b_2 y_2'}$$
$$\cancel{- b_1 y_1 a_1 y_1'} - b_1 y_1 a_2 y_2' - b_2 y_2 a_1 y_1' - \cancel{b_2 y_2 a_2 y_2'}$$
$$= a_1 b_2 y_1 y_2' - a_1 b_2 y_1' y_2 + a_2 b_1 y_1' y_2 - a_2 b_1 y_1 y_2'$$

$(a_1 b_2 - a_2 b_1) W[y_1, y_2]$
$= (a_1 b_2 - a_2 b_1)(y_1 y_2' - y_1' y_2)$
$= a_1 b_2 y_1 y_2' - a_1 b_2 y_1' y_2 - a_2 b_1 y_1 y_2' + a_2 b_1 y_1' y_2$

LHS = RHS

9. Solve $y'' - 2y' + 2y = 0$

Soln:
$r^2 - 2r + 2 = 0$
$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \dfrac{2 \pm \sqrt{4-8}}{2}$
$= \dfrac{2 \pm 2i}{2}$
$= 1 \pm i$
$\lambda = 1, \; u = 1 \rightarrow y_1 = e^{\lambda t} \cos(ut), \; y_2 = e^{\lambda t} \sin(ut)$

$y = c_1 e^t \cos t + c_2 e^t \sin t$

10. Solve $y'' - 2y' + 6y = 0$

Soln:
$r^2 - 2r + 6 = 0$
$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \dfrac{2 \pm 2i\sqrt{5}}{2}$
$= 1 + i\sqrt{5}$
$\lambda = 1, \; u = \sqrt{5}$

$y = c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t)$